Anomalous diffusion and weak chaos in a classical Bose-Hubbard chain

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Introduction and setup

We analyze one-dimensional Bose-Hubbard chain with L sites in the semiclassical regime with total occupation number $N \to \infty$. This system is nonintegrable and exhibits both classical and quantum chaos. The quantum and semiclassical Hamiltonians of the system read:

$$H_{\mathrm{BH}} = \sum_{j=1}^{L}{}' \left[-\frac{J}{2}\left(b_j^\dagger b_{j+1} + b_j b_{j+1}^\dagger\right) + \frac{U_{\mathrm{BH}}}{2} n_j\left(n_j - 1\right) - \mu n_j \right] \longrightarrow H \equiv \frac{1}{N} H_{BH} = \sum_{j=1}^{L}{}' \left(\frac{U}{2} I_j^2 - \mu I_j\right) - J \sum_{j=1}^{L}{}' \left(\sqrt{I_j I_{j+1}} \cos\left(\phi_j - \phi_{j+1}\right)\right).$$

where J is the hopping strength, U_{BH} is the on-site Coulomb repulsion energy, μ is the chemical potential, $b_j^\dagger$, b_j are the bosonic creation and annihilation operators, n_j is the number operator at site j, $U \equiv U_{\mathrm{BH}} N$ is the rescaled Coulomb energy and I_j, ϕ_j are classical action-angle variables for this Hamiltonian. The actions I_j are really semiclassical expectation values of occupation numbers: $I_j = \langle b_j^\dagger b_j \rangle|_{N \to \infty}$.

Dynamics and weak chaos

As could be expected, the Mott vs. superfluid regime (U/J large vs. U/J small) is characterized by mostly localized vs. mostly delocalized classical orbits. Typical orbits are shown in Figure 1.

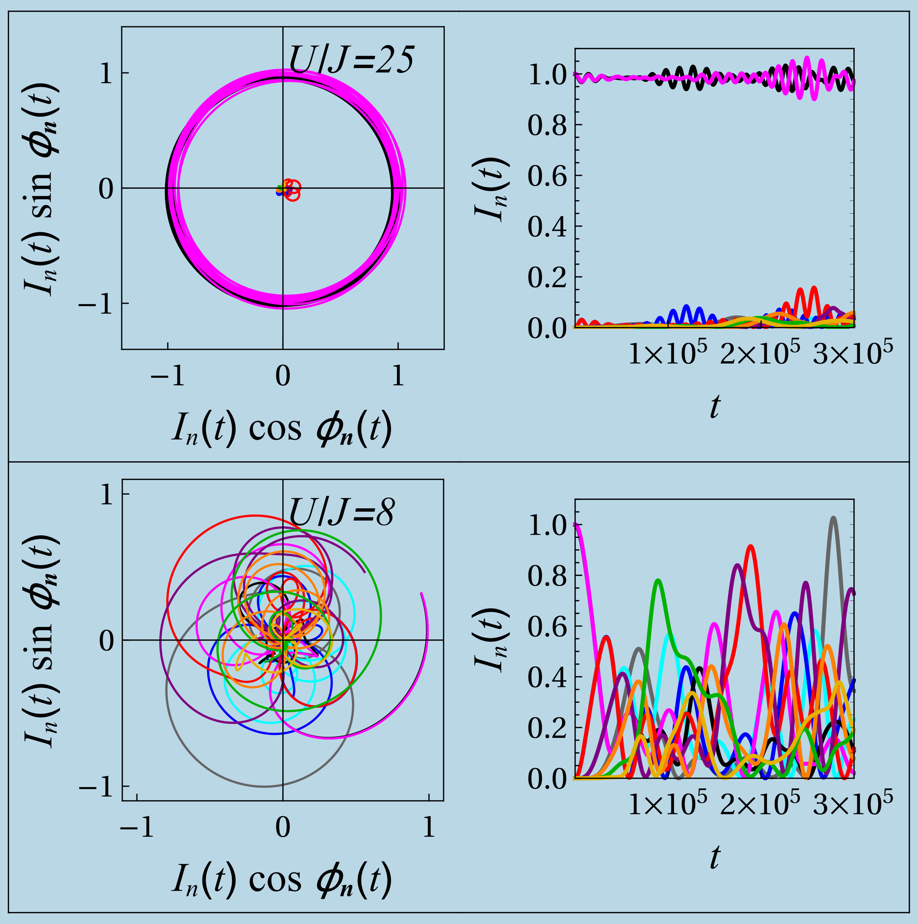

Figure 1: *The orbits (left) and the time evolution of actions (right) for the two Mott regime (top) and the superfluid regime (bottom). The chain length is $L = 10$.*

However, the existence of chaos is completely insensitive to the Mott/superfluid transition and solely depends on occupation numbers (Figure 2). Initially filled sites ($n = 4, 5$) always exhibit strong chaos and large Lyapunov exponents, whilst the dynamics of initially empty sites is to a good extent regular, even for chains of length $L \sim 100$ (despite the widespread belief that many-body nonintegrable systems typically show strong chaos).

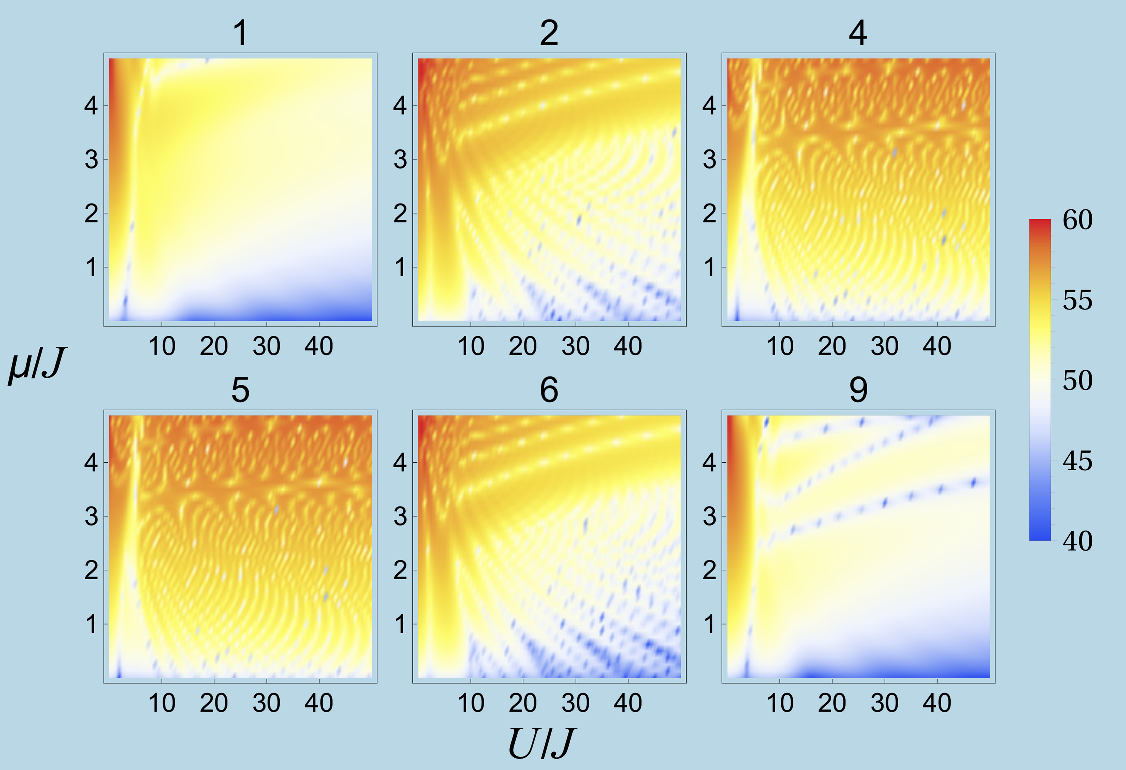

Figure 2: *Lyapunov exponent for the classical orbits corresponding to actions/occupation numbers at sites 1,2,4,5,6,9 as a function of chemical potential and Coulomb repulsion. Initially filled sites 4 and 5 show strong chaos.*

Anomalous and normal diffusion

This system exhibits anomalous diffusion (superdiffusion) in the sense that the growth of the variance of the action in some ensemble of orbits grows faster than linear: $\langle \Delta I^2 \rangle \sim t^\zeta, \zeta > 1$. ($\zeta = 1$ would be normal diffusion). This is true up to some crossover time t_0. The superdiffusion exponents take their values from two discrete sets:

- For site number n, the transport exponent is generically $\zeta_n = 4m$ ($m = 0, 1, 2 \ldots$), where m is the distance to the nearest non-empty site (if the site n itself starts non-empty we have $m = 0$).
- For specific initial combinations of occupation numbers with a blend of **resonant and non-resonant** sites, the transport exponent becomes $\zeta_n = 2m$ ($m = 0, 1, 2 \ldots$).

The diffusion of actions is nothing but particle transport – remember that actions I_j are the semiclassical values of the occupation numbers n_j. **The same exponents appear both in Mott and superfluid regimes**, as can be seen in Figure 3.

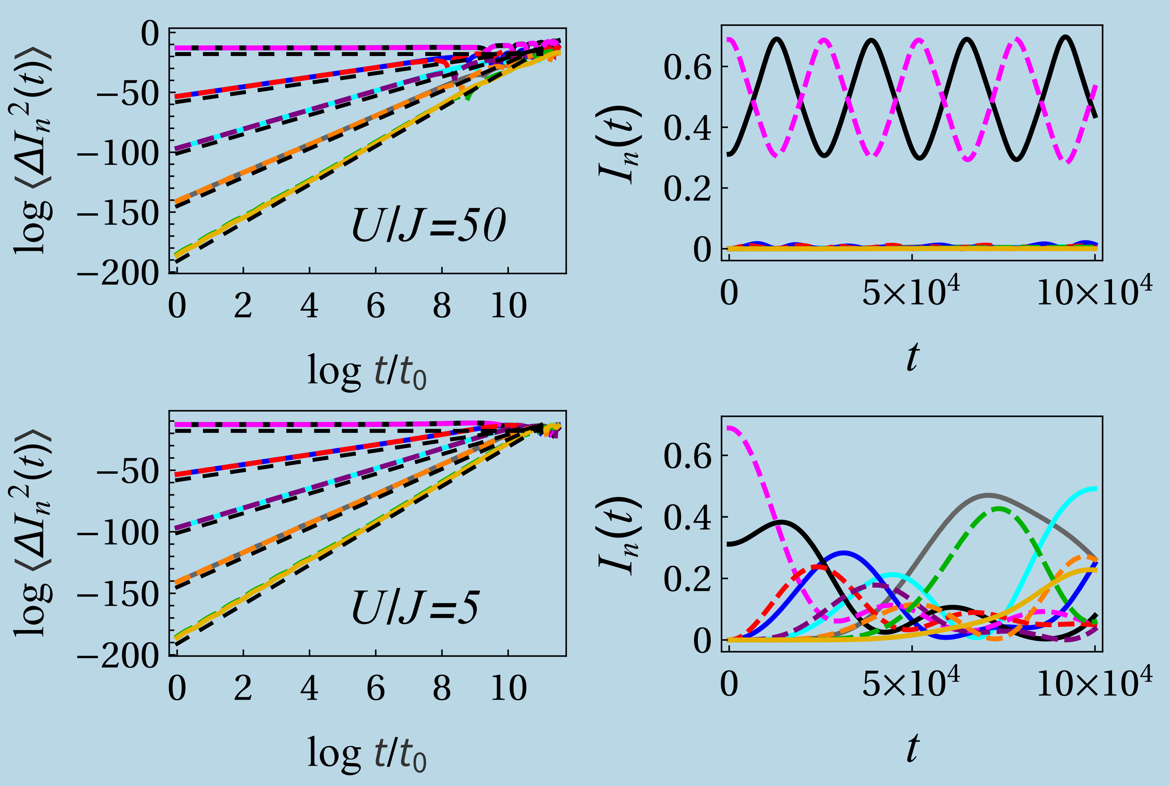

Figure 3: *Log-log plot of the action variance growth in time (left) and mean values of actions (right) in the Mott regime (top) and superfluid regime (bottom). Colors encode different sites and black dashed lines are the fits to analytic expressions $\langle \Delta I^2 \rangle \sim t^{4m}, m \in \mathbb{N}$.*

A very simple derivation of the superdiffusion exponents ζ_n can be obtained within the formalism of G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002). Perturbative expansion of the BH Hamiltonian near a filled site leads to a pendulum-like effective Hamiltonian:

$$H_{\mathrm{pert}} \sim \frac{U}{2} I_1^2 - \frac{\mathrm{const.}}{I_0} \cos\phi_1,$$

where $I_0 \sim 1$ is an initially filled site and I_1 is the initially empty neighboring site. The pendulum period is $T_1 \propto \sqrt{I_0}$, leading to the scale invariance in time and in action space:

$$(T_1, I_1) \mapsto (\lambda_T T_1, \lambda_I I_1), \ \lambda_I = \lambda_T^2.$$

According to Zaslavsky, the diffusion exponent for neighboring sites is $\zeta_1 = 2\log\lambda_I/\lambda_T = 4$. Iterating this for sites of distance m we get $\zeta_n = 4m$. The case $\zeta_m = 2m$ is more complicated as it depends sensitively on initial conditions.

At late times ($t \gg t_0$), the system effectively thermalizes and normal diffusion (hydrodynamic regime) replaces anomalous diffusion (Figure 4). In the early (superdiffusion) regime the partition function is well described by the **quenched** approximation (considering the actions as quenched) and in the late normal diffusion regime is becomes **annealed** in the actions.

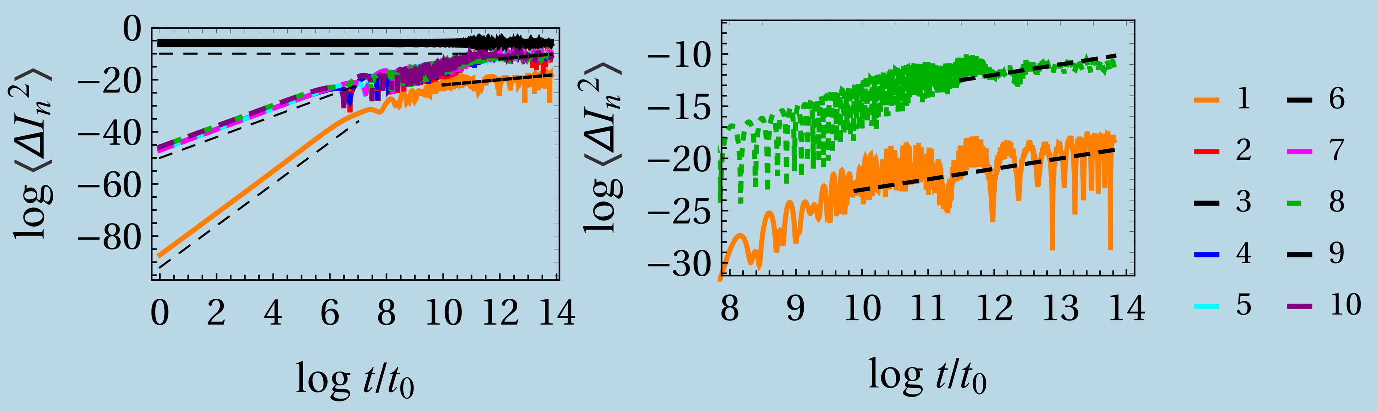

Figure 4: *Log-log plot of the action variance growth in time. In the left panel we see both superdiffusion (with exponents $\zeta_n = 0, 4, 8$) at early times and normal diffusion ($\zeta_n = 1$) for late times. In the right panel we zoom-in to the normal diffusion regime.*

The normal diffusion coefficient can be derived from the Langevin equation in the Ito formalism. Regarding the angle-dependent part of the equations of motion as a Wiener process, we write the Langevin equation as

$$dI_n = g_{nm}(\mathbf{I}(\mathbf{t} = \mathbf{0}))dW_m(t)$$
$$\langle W_i(t) W_j(t') \rangle = \delta(t - t')\sigma_{ij}^2,$$

where σ_{ij}^2 is obtained by averaging the action-dependent part of the equations of motion.

Reference

D. Marković and M. Čubrović, Chaos and anomalous transport in a semiclassical Bose-Hubbard chain, arXiv: **2308.14720**

Discrete nonlinear Schrödinger equation (DNSE)

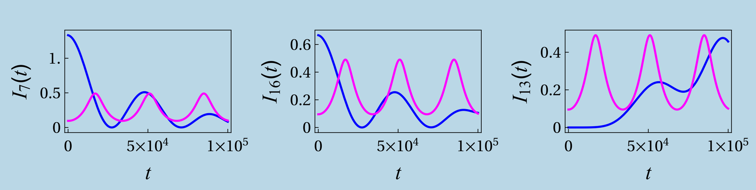

Figure 5: *Evolution of actions (blue) versus the analytic prediction from DNSE (magenta). The period of oscillations (2μ) is in very good agreement.*